Symmetry classes of alternating-sign matrices under one roof

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In a previous article [23], we derived the alternating-sign matrix (ASM) theorem from the Izergin-Korepin determinant [12, 13, 19] for a partition function for square ice with domain wall boundary. Here we show that the same argument enumerates three other symmetry classes of alternating-sign matrices: VSASMs (vertically symmetric ASMs), even HTSASMs (half-turn-symmetric ASMs), and even QTSASMs (quarter-turn-symmetric ASMs). The VSASM enumeration was conjectured by Mills; the others by Robbins [31]. We introduce several new types of ASMs: UASMs (ASMs with a U-turn side), UUASMs (two U-turn sides), OSASMs (off-diagonally symmetric ASMs), OOSASMs (off-diagonally symmetric), and UOSASMs (off-diagonally symmetric with U-turn sides). UASMs generalize VSASMs, while UUASMs generalize VHSASMs (vertically and horizontally symmetric ASMs) and another new class, VHPASMs (vertically and horizontally perverse). OSASMs, OOSASMs, and UOSASMs are related to the remaining symmetry classes of ASMs, namely DSASMs (diagonally symmetric), DASASMs (diagonally symmetric), and TSASMs (totally symmetric ASMs). We enumerate several of these new classes, and we provide several 2-enumerations and 3-enumerations.

Our main technical tool is a set of multi-parameter determinant and Pfaffian formulas generalizing the Izergin-Korepin determinant for ASMs and the Tsuchiya determinant for UASMs [39]. We evaluate specializations of the determinants and Pfaffians using the factor exhaustion method.

1. INTRODUCTION

An alternating-sign matrix (or ASM) is a matrix with entries 1, 0, and -1, such that the non-zero entries alternate in sign in each row and column, and such that the first and last non-zero entry in each row and column is 1. Mills, Robbins, and Rumsey [27] conjectured a formula for the number of ASMs of order n. This formula was first proved by Zeilberger in 1995 [40], and later the author found a different proof [23]:

Theorem 1 (Zeilberger). There are

$$A(n) = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}$$

 $n \times n$ ASMs.

Theorem 1 is part of a larger, unfinished structure in enumerative combinatorics, much of it conjectured by Robbins [31, 32]. The structure includes two types of relations between alternating-sign matrices and another class of combinatorial objects, plane partitions in boxes. (A plane partition in a box is an order ideal in the poset $[1,\ldots,a]\times[1,\ldots,b]\times[1,\ldots,c]$. They can be interpreted as a basis for the irreducible Weyl representation $V(c\lambda_a)$ of the Lie algebra $\mathfrak{sl}(a+b)$.) One relation is by analogy: The number of plane partitions in a given box is *round* (meaning a product of small factors, also called *smooth*), and so is the number in any given symmetry class. It is usually easy to conjecture an explicit product formula for round numbers. Likewise Robbins found that the number of ASMs in most (but not all!) symmetry classes also seems to be round. The other relation is equinumeration. Robbins also

found that there are the same numbers, both round and not, of many types of ASMs as there are other types of plane partitions. To begin with there are exactly as many ASMs with no symmetry as there are plane partitions with full symmetry, and this is what Zeilberger more directly proved.

Our proof of Theorem 1 sheds no light on plane partitions, but it does rely on a connection to another important structure in quantum algebra and statistical mechanics, the Yang-Baxter equation. Using the Yang-Baxter equation, Izergin and Korepin found a determinant formula for the partition function of square ice with domain wall boundary conditions [12, 13, 19]. We noted that this state model is equivalent to certain weighted enumerations of ASMs. Although the determinant is singular at the point where all weights are equal, it is generically non-singular and round along a special curve of weights with a coordinate q such that q = 1 is the equal-weight point. (As defined in Section 6 roundness of a polynomial in q is stronger than smoothness.) Finally the q-specialization of the Izergin-Korepin determinant independently generalizes to a round determinant in two parameters p and q. The twoparameter determinant can be evaluated by factor exhaustion

In this article we generalize this argument to some of the previously known classes of ASMs and also some new ones. The Izergin-Korepin determinant may have seemed accidental, but we find a similar formula for each symmetry class of ASMs after modifying the boundary conditions at the symmetry lines (Theorem 10). (One case was found previously by Tsuchiya [39].) The round q-specialization and its two-parameter generalization may also have seemed accidental, but we find many such specializations complemented by four two-parameter determinants and two three-parameter Pfaffians (Sections 6 and 7). Besides ordinary enumeration, the q-specializations also include several x-enumerations (in which each orbit of -1 entries in the ASM has weight x) with x=2 and x=3. (Along the way we will correct an error in the 3-enumeration of ASMs in Reference 23 found by Robin Chap-

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man.) Besides the q-specializations we establish that many of the x-enumerations divide each other or otherwise share large factors (Section 7).

We speculate that our constructions are part of a yet larger and possibly less exotic structure in quantum algebra. In particular the solution to the Yang-Baxter equation that we use corresponds to the 2-dimensional representation of the Lie algebra $\mathfrak{sl}(2)$. We have not investigated what happens when $\mathfrak{sl}(2)$ is replaced by another Lie algebra or the 2-dimensional representation by another representation.

Acknowledgments

The present work began with the mistake found in Reference 23 by Robin Chapman and with the Tsuchiya determinant, which is the UASM case of Theorem 10 and which was brought to the author's attention by Jim Propp. We would like to thank Vladimir Korepin, Robin Chapman, and Jim Propp more generally for their attention to the author's work. We would also like to thank Jan de Gier, Christian Krattenthaler, Soichi Okada, Neil Sloane, and Paul Zinn-Justin for their interest and for finding mistakes in earlier drafts. Finally we would like to acknowledge works by Bressoud [1], Bressoud and Propp [2], Robbins [32], and Zeilberger [40] for spurring the author's interest in alternating-sign matrices.

Mathematical experiments in Maple [26] were essential at every stage of this work. This article is typeset using REVTeX 4 [30] and PSTricks [29].

2. STATEMENT OF RESULTS

In general the *x-enumeration* of a class of ASMs is defined as the total *x*-weight. The *x*-weight of an ASM is x^n if it has *n* symmetry orbits of negative entries. Figure 1 shows an example. (In the figures, we use + and - for 1 and -1.) Note that the 2-enumerations of any class of ASMs is a special case (called the free fermion point in statistical mechanics) that can often be established by other methods [4, 5, 9, 10, 14, 15, 21, 22, 27, 28, 31, 33, 34]. The reader can also consider the elementary case of 0-enumeration since it is also often round.

$$\begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ + & 0 & - & + & - & 0 & + \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: A VSASM with x-weight x^2 .

We summarize the conventional symmetry classes of ASMs and what is known and conjectured about their basic enumerations.

- ASMs alternating-sign matrices. The 1-, 2-, and 3-enumerations are all previously known.
- VSASMs vertically symmetric ASMs. The 2-enumeration is previously known. Mills conjectured the 1-enumeration [32]. We establish the 1-, 2-, and 3-enumerations.
- HTSASMs half-turn symmetric ASMs. Robbins conjectured the 1-enumeration and established the 2-enumeration [31]. We establish the 1-, 2-, and 3-enumerations, but only for even order.
- QTASMs quarter-turn symmetric ASMs. Robbins conjectured the enumeration. We establish the 1- and 2-enumerations only for even order. We also prove a formula for a factor of the 3-enumeration; the other factor does not appear to be round.
- VHSASMs vertically and horizontally symmetric ASMs.
 Robbins conjectured the 1-enumeration and the 2-enumeration is previously known [15]. We only establish a determinant formula.
- DSASMs diagonally symmetric ASMs. Their number does not appear to be round and no determinant or Pfaffian formula is known.
- DASASMs diagonally and antidiagonally symmetric ASMs. Robbins conjectured the enumeration for odd order, but their number does not appear to be round for even order. No determinant or Pfaffian formula is known.
- TSASMs totally symmetric ASMs. Their number does not appear to be round and no determinant or Pfaffian formula is known.

Our results for these classes are as follows:

Theorem 2. The number of $n \times n$ ASMs is given by

$$A(n) = (-3)^{\binom{n}{2}} \prod_{i,j} \frac{3(j-i)+1}{j-i+n}.$$

The number of $2n + 1 \times 2n + 1$ vertically symmetric ASMs (VSASMs) is given by

$$A_{V}(2n+1) = (-3)^{n^{2}} \prod_{\substack{i,j \leq 2n+1\\2|j}} \frac{3(j-i)+1}{j-i+2n+1}.$$

The number of $2n \times 2n$ half-turn symmetric ASMs (HTSASMs) is given by

$$\frac{A_{\rm HT}(2n)}{A(n)} = (-3)^{\binom{n}{2}} \prod_{i,j} \frac{3(j-i)+2}{j-i+n}.$$

The number of $4n \times 4n$ quarter-turn symmetric ASMs (QT-SASMs) is given by

$$A_{\text{OT}}(4n) = A_{\text{HT}}(2n)A(n)^2.$$

In the statement of Theorem 2 and throughout this article, subscripts and products range from 1 to n unless otherwise specified.

Theorem 3. The 2- and 3-enumerations of ASMs and VSASMs are given by

$$\begin{split} A(n;2) &= 2^{\binom{n}{2}} \\ A(n;3) &= \frac{3^{n^2-n}}{2^{n^2-n}} \prod_{\substack{i,j \\ 2\nmid j-i}} \frac{3(j-i)+1}{3(j-i)} \\ A_{\mathrm{V}}(2n+1;2) &= 2^{n^2-n} \\ A_{\mathrm{V}}(2n+1;3) &= \frac{3^{2n^2}}{2^{2n^2+n}} \prod_{\substack{i,j \leq 2n+1 \\ 2\nmid i,2 \mid j}} \frac{3(j-i)+1}{3(j-i)}. \end{split}$$

The 2-enumerations of even-sized HTSASMs and QTSASMs are given by

$$\begin{split} A_{\mathrm{HT}}(2n;2,1) &= 2^{n^2} \prod_{\substack{i,j \\ 2 \nmid j-i}} \frac{2(j-i)+1}{2(j-i)} \\ A_{\mathrm{QT}}(4n;2) &= (-1)^{\binom{n}{2}} 2^{2n^2-n} \prod_{i,j} \frac{4(j-i)+1}{j-i+n}. \end{split}$$

$$\begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & 0 & - & 0 & + & 0 \\ + & - & + & * & + & - & + \\ 0 & + & 0 & - & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$$

Figure 2: The simplest VHPASM.

$$\begin{pmatrix}
0 & 0 & + \\
0 & + & - \\
+ & - & 0 \\
0 & 0 & + \\
0 & + & 0 \\
0 & 0 & 0
\end{pmatrix}.$$

Figure 3: A UASM.

We will also consider the following new types of ASMs:

• VHPASMs¹ - vertically and horizontally perverse ASMs. A VHPASM has dimensions $4n + 1 \times 4n + 3$ for some

integer *n*. It satisfies the alternating-sign condition and it has the same symmetries as a VHSASM, except that the central entry (*) has the opposite sign when read horizontally as when read vertically. The simplest VH-PASM is given in Figure 2.

- UASMs $2n \times 2n$ ASMs with U-turn boundary on the right. Figure 3 shows an example of a UASM. As the example indicates, a UASM is vertically just like an ASM. Horizontally the signs alternate if we read the 2k-1st row from left to right, and then continue to alternate if we read the 2kth row from right to left. UASMs were first considered by Tsuchiya [39]. They generalize VSASMs.
- UUASMs 2 $2n \times 2n$ ASMs with U-turn boundary on the top and right. UUASMs generalize both VHSASMs and VHPASMs.
- OSASMs off-diagonally symmetric ASMs. *I.e.*, DSASMs with a null diagonal.
- OOSASMs off-diagonally, off-antidiagonally symmetric ASMs. I.e., DASASMs with null diagonals.
- UOSASMs off-diagonally symmetric UASMs. They include TSASMs with null diagonals.

Our remaining results involve weighted enumerations that are more general than just x-enumeration. We define the yweight of a $2n \times 2n$ HTSASM to be y^k if the HTSASM has k non-zero entries in the upper left quadrant. This yields the (x,y)-enumeration of HTSASMs, which is round when y is -1 and x is 1 or 3.) We define the x-weight of a UASM or a UUASM be the number of -1s, as before. We define the y-weight of a UASM to be y^k if k of the U-turns are oriented upward in the corresponding square ice state. We define the y-weight of a UUASM the same way using the U-turns on the right, and define the z-weight of a UUASM to be z^k if k of the U-turns on the top are oriented to the right. Thus we can consider the (x, y)-enumeration of UASMs and the (x, y, z)-weight of UUASMs. As with UASMs, the y-weight of a UOSASM is y^k if k of the U-turns on the top are oriented to the right. By contrast, the y-weight of an OOSASM is y^k if there are 2kmore 1s than -1s in the upper left quadrant. Finally we index the generating function of a given type of ASM by the length of one of its rows, counting the length twice if the row takes a U-turn, and we include x-, y-, and z-weight where applicable. For example $A_{\text{LIO}}(8n; x, y)$ is the weighted number of $4n \times 4n$ UOSASMs.

 $^{^{1}}$ Also known as β -ASMs, since their boundary conditions are incompatible with VHS.

² Also known as Unix-to-Unix ASMs.

Theorem 4. There exist polynomials satisfying the equations

$$\begin{split} A(2n;x) &= 2A_{\rm V}(2n+1;x)\widetilde{A}_{\rm V}(2n;x) \\ A(2n+1;x) &= A_{\rm V}(2n+1;x)\widetilde{A}_{\rm V}(2n+2;x) \\ A_{\rm U}(2n;x,y) &= (y+1)^nA_{\rm V}(2n+1;x) \\ A_{\rm HT}(2n;x,\pm 1) &= A(n;x)A_{\rm HT}^{(2)}(2n;x,\pm 1) \\ A_{\rm UU}(4n;x,y,z) &= A_{\rm V}(2n+1;x)A_{\rm UU}^{(2)}(4n;x,y,z) \\ A_{\rm OO}(4n;x,y) &= A_{\rm O}(2n;x)A_{\rm OO}^{(2)}(4n;x,y) \\ A_{\rm QT}(4n;x) &= A_{\rm UO}^{(1)}(4n;x)A_{\rm QT}^{(2)}(4n;x) \\ A_{\rm UO}(8n;x,y) &= A_{\rm UO}^{(1)}(8n;x)A_{\rm UO}^{(2)}(8n;x,y) \\ A_{\rm HT}^{(2)}(4n;x,1) &= A_{\rm UU}^{(2)}(4n;x,1,1)\widetilde{A}_{\rm UU}^{(2)}(4n;x) \\ A_{\rm HT}^{(2)}(4n;x,1) &= 2A_{\rm UU}^{(2)}(4n;x,1,1)\widetilde{A}_{\rm UU}^{(2)}(4n+4;x) \\ A_{\rm HT}^{(2)}(4n;x,-1) &= (-x)^nA_{\rm UU}^{(1)}(8n;x,1)\widetilde{A}_{\rm UO}^{(1)}(8n;x,1). \end{split}$$

Many of the factorizations in Theorem 4 were conjectured experimentally by David Robbins [31]; the formula for $A_{\rm U}(2n;x,y)$ was conjectured by Cohn and Propp [6].

Theorem 5. The generating functions in Theorem 4 have the following special values.

$$\begin{split} A_{\rm O}(2n) &= A_{\rm V}(2n+1) \\ A_{\rm UU}^{(2)}(4n;1,1,1) &= (-3)^{n^2} 2^{2n} \prod_{\substack{i,j \leq 2n+1 \\ 2 \mid j}} \frac{3(j-i)+2}{j-i+2n+1} \\ A_{\rm UU}^{(2)}(4n;2,1,1) &= 2^{n(n+2)} \prod_{\substack{i,j \leq 2n+1 \\ 2 \nmid i,2 \mid j}} \frac{2(j-i)+1}{2(j-i)} \\ A_{\rm VHP}^{(2)}(4n+2;1) &= A_{\rm V}(2n+1) \\ A_{\rm QT}^{(1)}(4n) &= A(n)^2 \\ A_{\rm QT}^{(1)}(4n;2) &= (-1)^{\binom{n}{2}} 2^{n(n-1)} \prod_{\substack{i,j}} \frac{4(j-i)+1}{j-i+n} \\ A_{\rm QT}^{(1)}(4n;3) &= 3^{\binom{n}{2}} A(n) \\ A_{\rm UO}^{(1)}(8n) &= A_{\rm V}(2n+1)^2 \\ A_{\rm UO}^{(2)}(8n) &= A_{\rm UU}(4n). \end{split}$$

Other identities, for example that

$$A_{\text{OT}}^{(2)}(4n) = A_{\text{HT}}(2n),$$

are implied by combining Theorems 2, 3, 4, and 5, although such combinations do not always reflect the logic of the proofs.

3. SOUARE ICE

If G is a tetravalent graph, an *ice state* (also called a *six-vertex state*) of G is an orientation of the edges such that two

edges enter and leave every tetravalent vertex. In particular if G is locally a square grid, then the set of ice states is called *square ice* [25]. More generally G may also have some univalent vertices, which are called *boundary*, and restrictions on the orientations of the boundary edges are called *boundary conditions*.

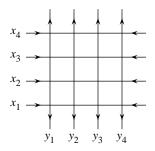


Figure 4: Square ice with domain wall boundary.

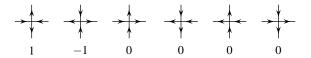


Figure 5: Replacing square ice with alternating-sign entries.

For example, a finite square region of square ice can have *domain wall boundary*, defined as in at the sides and out at the top and bottom, as in Figure 4. These boundary conditions were first considered by Korepin [12, 18, 19]. A square ice state on this region yields a matrix if we replace each vertex by a number according Figure 5. It is easy to check that this transformation is a bijection between square ice with domain wall boundary and alternating-sign matrices [10, 23].

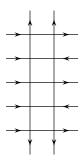


Figure 6: Square ice with VS boundary.

There are also easy bijections from ice states of the graphs in Figures 6, 7, 8, and 9 to the sets of VSASMs, VHSASMs, even HTSASMs, and even QTSASMs. (The labels in these figures will be used later.) The dashed line in the QTSASM graph means that the orientation of an edge reverses as it

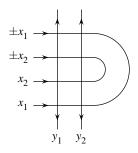


Figure 7: Square ice with HTS boundary.

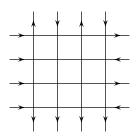


Figure 8: Square ice with VHS boundary.

crosses the line. The HTSASM and QTSASM graphs are obtained by quotienting the unrestricted ASM graph by the symmetry. The median of a $2n+1\times 2n+1$ VSASM is always the same, so we can delete it and consider the alternating-sign patterns on the left half. The deleted median then produces the alternating boundary in Figure 6. Likewise we can quarter a VHSASM by deleting both medians, which produces two alternating sides.

Finally the square ice grids corresponding to UASMs, UUASMs, OSASMs, OOSASMs, and UOSASMs are shown

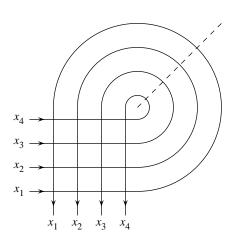


Figure 9: Square ice with QTS boundary.

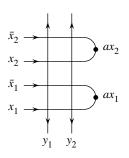


Figure 10: Square ice with U boundary.

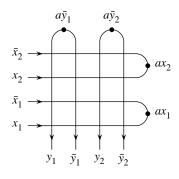


Figure 11: Square ice with UU boundary.

in Figures 10, 11, 12, 13, and 14. The last three grids have right-angled divalent vertices; we require the orientations of a square ice state to either be both in or both out at these vertices. In contrast at the U-turn vertices one edge must point in and one must point out.

4. LOCAL CONCERNS

Throughout the article we assume the following abbreviations:

$$\bar{x} = x^{-1}$$

$$\sigma(x) = x - \bar{x}$$

$$\alpha(x) = \sigma(ax)\sigma(a\bar{x}).$$

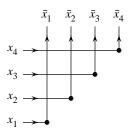


Figure 12: Square ice with OS boundary.

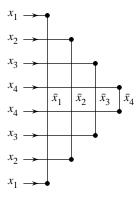


Figure 13: Square ice with OOS boundary.

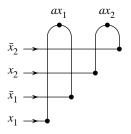


Figure 14: Square ice with UOS boundary.

(As we discuss below, a is a global parameter that need not appear as an explicit argument of α .)

We will consider a class of multiplicative weights for symmetric ASMs. By a multiplicative weight we mean that the weight of some object is the product of the weights of its parts. In statistical mechanics, multiplicative weights are called Boltzmann weights, and the total weight of all objects is called a partition function. Figure 15 shows the weights that we will use for the six possible states of a vertex. The figure also shows the weights for U-turns and corners that are labelled with a dot; bare edges and curves have the trivial weight 1. The vertex weights are called an *R-matrix* and the U-turn and corner weights are called K-matrices. Vertex and U-turn weights depend on a parameter x (the spectral parameter) which may be different for different vertices or Uturns, so we will label sites by the value of x. The weights also depend on three parameters a, b, and c which will be the same for all elements of any single square ice grid, so these parameters do not appear as labels.

We will use a graph with labelled vertices as a notation for its corresponding partition function. If the graph has unoriented boundary edges, then the partition function is also interpreted as a function of the orientations of the edges. On the other hand, our definitions imply that we sum over the orien-

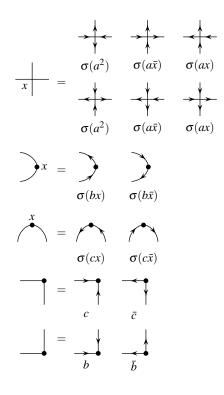


Figure 15: Weights for vertices and U-turns.

tations of internal edges. For example, the graph



denotes the following function on the set of four orientations of the boundary:

In this notation a vertex is not quite invariant under rotation by 90 degrees, so the meaning of a label depends on the quadrant in which it appears. The following relation holds:

$$\frac{}{x} = \frac{}{\bar{x}}$$

As a further abbreviation, if we label two lines of a graph that cross at an unlabelled vertex, the spectral parameter is set to their ratio:

$$x - \frac{1}{|y|} = \frac{1}{|x\bar{y}|}$$

The labelled graphs in Figures 4, 6, 7, 9, 10, 11, 12, 13, and 14 then represent the partition functions

$$\begin{array}{cccc} Z(n;\vec{x},\vec{y}) & Z_{\mathrm{HT}}^{\pm}(n;\vec{x},\vec{y}) & Z_{\mathrm{U}}(n;\vec{x},\vec{y}) & Z_{\mathrm{UU}}(n;\vec{x},\vec{y}) \\ Z_{\mathrm{OT}}(n;\vec{x}) & Z_{\mathrm{OO}}(n;\vec{x}) & Z_{\mathrm{OO}}(n;\vec{x}) & Z_{\mathrm{UO}}(n;\vec{x}) \end{array}$$

Here the vectors \vec{x} and \vec{y} have length n when both are present, and otherwise \vec{x} has length 2n. In the HT and OO cases there is a single extra parameter taken from the set $\{+,-\}$; if it is — then the spectral parameters in the upper half of the grid are negated. (Note that the index n is not defined in the same way as for the enumerators such as $A_{\rm HT}(2n)$.)

The key property of the *R*-matrix is that it satisfies the Yang-Baxter equation:

Lemma 6 (Yang-Baxter equation). *If* $xyz = \bar{a}$, then

$$z$$
 $=$ x z z z z z

As usual the Yang-Baxter equation appears to be a massive coincidence. In our previous review of the Yang-Baxter equation [23], the R-matrix was normalized to have a particular symmetry: It was the matrix of an invariant tensor over the 2-dimensional representation of the quantum group $U_q(\mathfrak{sl}(2))$, with q related to our present parameter a. This symmetry reduced the coincidence in the equation to a single numerical equality. The spectral parameters were chosen to satisfy the equality. Here we normalize the R-matrix to reveal combinatorial symmetry rather than symmetry from quantum algebra.

Proof. Taken literally, the equation consists of 64 numerical equalities, because there are 64 ways to orient the six boundary edges on each side. However, both sides are zero unless three edges point in and three point out. This leaves 20 nonzero equations. The equation also has three kinds of symmetry: The right side is the left side rotated by 180 degrees, all arrows may be reversed, and both sides may be rotated by 120 degrees if the variables x, y, and z are cyclically permuted. By the three symmetries, 8 of the non-zero equations are tautological, and the other 12 are all equivalent. One of the 12 non-trivial equations is

$$z$$
 $=$ x z z z z z

In algebraic form, the equation is

$$\sigma(a\bar{y})\sigma(a^2)\sigma(a\bar{x}) = \sigma(az)\sigma(a^2)^2 + \sigma(ax)\sigma(ay)\sigma(a^2).$$

Cancelling a factor of $\sigma(a^2)$, expanding, and cancelling terms yields

$$a^2 \bar{x} \bar{y} + \bar{a}^2 x y = a^3 z - z \bar{a} - \bar{z} a + \bar{a}^3 \bar{z} + a^2 x y + \bar{a}^2 \bar{x} \bar{y},$$

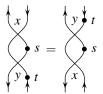
which is implied by the condition $xyz = \bar{a}$.

We will need the reflection equation [3, 7, 35], an analogue of the Yang-Baxter equation that relates a *K*-matrix to the *R*-matrix.

Lemma 7 (Reflection equation). *If* st = ay *and* $s\bar{t} = ax$, *then*

$$\begin{array}{c|c} x \\ \hline \\ y \\ t \end{array} = \begin{array}{c|c} y \\ t \\ \hline \\ x \\ \end{array}$$

Proof. The argument is similar to that for Lemma 6. Both sides are zero unless two boundary edges point in and two point out. There is a symmetry exchanging the two sides given by reflecting through a horizontal line and simultaneously reversing all arrows. (Note that the weights of a U-turn are not invariant under reflection alone.) Under this symmetry 4 of the 6 non-zero equations are tautological, and the other 2 are equivalent. One of these is:



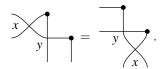
Algebraically, the equation reads:

$$\sigma(bt)\sigma(a^2)(\sigma(ay)\sigma(b\bar{s}) + \sigma(ax)\sigma(bs)) = \sigma(b\bar{t})\sigma(a^2)(\sigma(ax)\sigma(b\bar{s}) + \sigma(ay)\sigma(bs)).$$

All terms of the equation match or cancel when st = ay and $s\bar{t} = ax$.

The corner K-matrix also satisfies the reflection equation [7].

Lemma 8. For any x and y,



Proof. Diagonal reflection exchanges the two sides. Both sides are zero if an odd number of boundary edges point inward. If two boundary edges point in and the other two point out, then arrow reversal is also a symmetry, because one corner must have inward arrows and the other outward arrows. These facts together imply that all cases of the equation are null or tautological.

Finally we will need an equation that, loosely speaking, inverts a U-turn:

Lemma 9 (Fish equation). For any a and x,

$$\bar{a}x^2$$
 $ax = \sigma(a^2x^2)$ $a\bar{x}$

The proof is elementary.

5. DETERMINANTS

In this section we will establish determinant and Pfaffian formulas for the partition functions defined in Sections 3 and 4. Recall that the Pfaffian of an antisymmetric $2n \times 2n$ matrix A is defined as

$$\operatorname{Pf} A \stackrel{\text{def}}{=} \sum_{\pi \in X} (-1)^{\pi} \prod_{i} A_{\pi(2i-1), \pi(2i)},$$

where $X \subset S_{2n}$ has one representative in each coset of the wreath product $S_2 \wr S_n$. (Thus X admits a bijection with the set of perfect matchings of $\{1, \ldots, 2n\}$.) Recall also that

$$\det A = (\operatorname{Pf} A)^2$$
.

Theorem 10. Let

$$\begin{split} M(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\alpha(x_i\bar{y}_j)} \\ M^{\pm}_{\text{HT}}(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\sigma(a\bar{x}_iy_j)} \pm \frac{1}{\sigma(ax_i\bar{y}_j)} \\ M_{\text{U}}(n;\vec{x},\vec{y})_{i,j} &= \frac{1}{\alpha(x_i\bar{y}_j)} - \frac{1}{\alpha(x_iy_j)} \\ M_{\text{UU}}(n;\vec{x},\vec{y})_{i,j} &= \frac{\sigma(b\bar{y}_j)\sigma(cx_i)}{\sigma(ax_i\bar{y}_j)} - \frac{\sigma(b\bar{y}_j)\sigma(c\bar{x}_i)}{\sigma(a\bar{x}_i\bar{y}_j)} \\ &- \frac{\sigma(by_j)\sigma(cx_i)}{\sigma(ax_iy_j)} + \frac{\sigma(by_j)\sigma(c\bar{x}_i)}{\sigma(a\bar{x}_iy_j)} \\ M^{(k)}_{\text{QT}}(n;\vec{x})_{i,j} &= \frac{\sigma(\bar{x}_i^kx_j^k)}{\alpha(\bar{x}_ix_j)} \\ M_{\text{O}}(n;\vec{x})_{i,j} &= \frac{\sigma(\bar{x}_i^kx_j^k)}{\alpha(x_ix_j)} \\ M_{\text{OO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j) \left(\frac{c^2}{\sigma(ax_ix_j)} + \frac{b^2}{\sigma(a\bar{x}_i\bar{x}_j)}\right) \\ M^{(1)}_{\text{UO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j)\sigma(x_ix_j) \left(\frac{1}{\alpha(x_ix_j)} - \frac{1}{\alpha(\bar{x}_ix_j)}\right) \\ M^{(2)}_{\text{UO}}(n;\vec{x})_{i,j} &= \sigma(\bar{x}_ix_j)\sigma(x_ix_j) \left(\frac{\sigma(cx_i)\sigma(cx_j)}{\sigma(ax_ix_j)} - \frac{\sigma(cx_i)\sigma(c\bar{x}_j)}{\sigma(ax_i\bar{x}_j)}\right) \\ &- \frac{\sigma(c\bar{x}_i)\sigma(cx_j)}{\sigma(a\bar{x}_ix_i)} + \frac{\sigma(c\bar{x}_i)\sigma(c\bar{x}_j)}{\sigma(a\bar{x}_i\bar{x}_i)} \right). \end{split}$$

Then

$$\begin{split} Z(n; \vec{x}, \vec{y}) &= \frac{\sigma(a^2)^n \prod_{i,j} \alpha(x_i \bar{y}_j)}{\prod_{i < j} \sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j)} (\det M) \\ Z^{\pm}_{\mathrm{HT}}(n; \vec{x}, \vec{y}) &= \frac{\sigma(a^2)^n \prod_{i,j} \alpha(x_i \bar{y}_j)^2}{\prod_{i < j} \sigma(\bar{x}_i x_j)^2 \sigma(y_i \bar{y}_j)^2} (\det M) (\det M^{\pm}_{\mathrm{HT}}) \\ Z_{\mathrm{U}}(n; \vec{x}, \vec{y}) &= \frac{\sigma(a^2)^n \prod_i \sigma(b \bar{y}_i) \sigma(a^2 x_i^2) \prod_{i,j} \alpha(x_i \bar{y}_j) \alpha(x_i y_j)}{\prod_{i < j} \sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j) \prod_{i \le j} \sigma(\bar{x}_i \bar{x}_j) \sigma(y_i y_j)} \\ & \cdot (\det M_{\mathrm{H}}) \end{split}$$

$$\begin{split} Z_{\text{UU}}(n; \vec{x}, \vec{y}) &= \frac{\sigma(a^2)^n \prod_i \sigma(a^2 x_i^2) \sigma(a^2 \vec{y}_i^2) \prod_{i,j} \alpha(x_i \vec{y}_j)^2 \alpha(x_i y_j)^2}{\prod_{i < j} \sigma(\bar{x}_i x_j)^2 \sigma(y_i \vec{y}_j)^2 \prod_{i \le j} \sigma(\bar{x}_i \vec{x}_j)^2 \sigma(y_i y_j)^2} \\ & \cdot (\det M_{\text{U}}) (\det M_{\text{UU}}) \\ Z_{\text{QT}}(n; \vec{x}) &= \frac{\sigma(a^2)^n \sigma(a)^{3n} \prod_{i < j \le 2n} \alpha(\bar{x}_i x_j)^2}{\prod_{i < j \le 2n} \sigma(\bar{x}_i x_j)^2} \\ & \cdot (\operatorname{Pf} M_{\text{QT}}^{(1)}) (\operatorname{Pf} M_{\text{QT}}^{(2)}) \\ Z_{\text{O}}(n; \vec{x}) &= \frac{\sigma(a^2)^n \prod_{i < j \le 2n} \alpha(x_i x_j)}{\prod_{i < j \le 2n} \sigma(\bar{x}_i x_j)} (\operatorname{Pf} M_{\text{O}}) \\ Z_{\text{OO}}(n; \vec{x}) &= \frac{\sigma(a^2)^n \prod_{i < j \le 2n} \alpha(x_i x_j)^2}{\prod_{i < j \le 2n} \sigma(\bar{x}_i x_j)^2} (\operatorname{Pf} M_{\text{O}}) (\operatorname{Pf} M_{\text{OO}}) \\ Z_{\text{UO}}(n; \vec{x}) &= \frac{b^{2n} \sigma(a^2)^n \sigma(a)^{2n} \prod_{i \le j \le 2n} \sigma(a^2 x_i^2)}{\prod_{i < j \le 2n} \sigma(\bar{x}_i x_j)^2 \prod_{i \le j \le 2n} \sigma(x_i x_j)^2} \\ & \cdot \prod_{i < j \le 2n} \alpha(\bar{x}_i x_j)^2 \alpha(x_i x_j)^2 (\operatorname{Pf} M_{\text{UO}}^{(1)}) (\operatorname{Pf} M_{\text{UO}}^{(2)}) \end{split}$$

We call the first four partition functions the *determinant* partition functions and the other four the *Pfaffian* partition functions.

Remark. The partition function $Z_{\rm U}(n;\vec{x},\vec{y})$, the Tsuchiya determinant, is nearly invariant if \vec{x} is exchanged with \vec{y} . Similarly $Z_{\rm O}(n;\vec{x})$ is nearly invariant if each x_i is replaced with \bar{x}_i . We have no direct explanation for these symmetries. Note that the first symmetry is less apparent in Tsuchiya's matrix M [39, eq. (42)], which has an asymmetric factor

$$F_{ij} = \frac{\sinh(\zeta_- + \lambda_j)}{\sinh(\lambda_i + \omega_i)} + \frac{\sinh(\zeta_- - \lambda_j)}{\sinh(\lambda_i - \omega_i)}.$$

In this expression ω_i , λ_j , and ζ_- are obtained from x_i , y_j , and b by reparameterization. If we factor this expression,

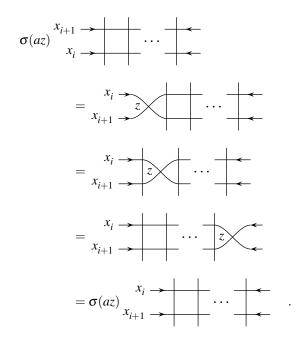
$$F_{ij} = \frac{\sinh(2\lambda_j)\sinh(\zeta_- - \omega_i)}{\sinh(\lambda_j + \omega_i)\sinh(\lambda_j - \omega_i)}.$$

we can then pull the asymmetric factors out of the determinant since they each depend on only one of the two indices i and j. This also explains why the K-matrix parameter ζ_- or b need not appear in the matrix M_U .

The proof of Theorem 10 uses recurrence relations that determine both sides. The relations are expressed in Lemmas 11, 12, 13, and 14. Indeed, the first three of these lemmas are obvious for the right-hand sides of Theorem 10; only Lemma 14 needs to be argued for both sides.

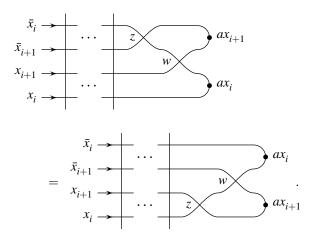
Lemma 11 (Baxter, Sklyanin). Each of the partition functions in Theorem 10 is symmetric in the coordinates of \vec{x} . Each determinant partition function is symmetric in the coordinates of \vec{y} . The partition functions $Z_U(n;\vec{x},\vec{y})$ and $Z_{UU}(n;\vec{x},\vec{y})$ gain a factor of $\sigma(a^2\vec{x}_i^2)/\sigma(a^2x_i^2)$ if x_i is replaced by \bar{x}_i for a single i. Similarly $Z_{UU}(n;\vec{x},\vec{y})$ gains $\sigma(a^2y_i^2)/\sigma(a^2\bar{y}_i^2)$ under $y_i\mapsto\bar{y}_i$ and $Z_{UO}(n;\vec{x})$ gains $\sigma(a^2\bar{x}_i^2)/\sigma(a^2x_i^2)$ under $x_i\mapsto\bar{x}_i$.

Proof. Invariance of $Z(n; \vec{x}, \vec{y})$ is an illustrative case. We exchange x_i with x_{i+1} for any $i \le n-1$ by crossing the corresponding lines at the left side. If the spectral parameter of the crossing is $z = \bar{a}x_i\bar{x}_{i+1}$, we can move it to the right side using the Yang-Baxter equation (Lemma 6) and then remove it:



The argument for symmetry in \vec{x} is exactly the same for all of the square ice grids without U-turns. If the grid has diagonal boundary with corner vertices, we can bounce the crossing off of it using Lemma 8.

If the grid has U-turn boundary on the right, we exchange x_i with x_{i+1} by crossing the \bar{x}_i line over the two lines above it. We let the spectral parameters of these two crossings be $z = \bar{a}\bar{x}_ix_{i+1}$ and $w = \bar{a}\bar{x}_i\bar{x}_{i+1}$. We move both crossings to the right using the Yang-Baxter equation, then we bounce them off of the U-turns using the reflection equation (Lemma 7):



Also if the grid has a U-turn on the right, we establish covariance under $x_i \mapsto \bar{x}_i$ by switching the lines with these two labels and eating the crossing using the fish equation (Lemma 9).

The same arguments establish symmetry in the coordinates of \vec{y} . All of the arguments used in combination establish the claimed properties of $Z_{\text{UO}}(n;\vec{x})$.

Lemma 12. The partition function $Z_{\text{HT}}(n; \vec{x}, \vec{y})$ gains a factor of $(\pm 1)^n$ if x_i and y_i are replaced by \bar{x}_i and \bar{y}_i for all i simultaneously. Similarly $Z_{\text{OO},\pm}(n; \vec{x})$ is invariant under $x_i \mapsto \bar{x}_i$ and $b \mapsto c \mapsto b$.

Proof. In both cases, the symmetry is effected by reflecting the square ice grid or the alternating-sign matrices through a horizontal line.

For a vector $\vec{x} = (x_1, \dots, x_n)$, let $\vec{x}' = (x_2, \dots, x_n)$.

Lemma 13. *If* $x_1 = ay_1$, *then*

$$\begin{split} \frac{Z(n;\vec{x},\vec{y})}{Z(n-1;\vec{x}',\vec{y}')} &= \sigma(a^2) \prod_{2 \leq i} \sigma(a\bar{x}_i y_1) \sigma(a\bar{x}_1 y_i) \\ \frac{Z_{\rm HT}^{\pm}(n;\vec{x},\vec{y})}{Z_{\rm HT}^{\pm}(n-1;\vec{x}',\vec{y}')} &= \pm \sigma(a^2)^2 \prod_{2 \leq i} \sigma(a\bar{x}_i y_1)^2 \sigma(a\bar{x}_1 y_i)^2 \\ \frac{Z_{\rm U}(n;\vec{x},\vec{y})}{Z_{\rm U}(n-1;\vec{x}',\vec{y}')} &= \sigma(a^2) \sigma(a^2 x_1^2) \sigma(b\bar{y}_1) \\ & \cdot \prod_{2 \leq i} \sigma(a\bar{x}_i y_1) \sigma(a\bar{x}_1 y_i) \sigma(ax_i y_1) \sigma(a\bar{x}_1 \bar{y}_i) \\ \frac{Z_{\rm UU}(n;\vec{x},\vec{y})}{Z_{\rm UU}(n-1;\vec{x}',\vec{y}')} &= \sigma(a^2)^2 \sigma(a^2 x_1^2) \sigma(a^2 \bar{y}_1^2) \sigma(b\bar{y}_1) \sigma(cx_1) \\ & \cdot \prod_{2 \leq i} \sigma(a\bar{x}_i y_1)^2 \sigma(a\bar{x}_1 y_i)^2 \\ & \cdot \prod_{2 \leq i} \sigma(ax_i y_1)^2 \sigma(a\bar{x}_1 \bar{y}_i)^2. \end{split}$$

If $x_2 = ax_1$, then

$$\frac{Z_{\mathrm{QT}}(n;\vec{x})}{Z_{\mathrm{QT}}(n-1;\vec{x}'')} = \sigma(a)^2 \sigma(a^2)^2 \prod_{3 \leq i \leq 2n} \sigma(ax_i\bar{x}_1)^2 \sigma(a\bar{x}_ix_2)^2.$$

If $x_2 = \bar{a}\bar{x}_1$, then

$$\begin{split} \frac{Z_{\mathrm{O}}(n;\vec{x})}{Z_{\mathrm{O}}(n-1;\vec{x}'')} &= \sigma(a^2) \prod_{3 \leq i \leq 2n} \sigma(a\bar{x}_1 \bar{x}_i) \sigma(a\bar{x}_2 \bar{x}_i) \\ \frac{Z_{\mathrm{OO}}(n;\vec{x})}{Z_{\mathrm{OO}}(n-1;\vec{x}'')} &= c^2 \sigma(a^2)^2 \prod_{3 \leq i \leq 2n} \sigma(a\bar{x}_1 \bar{x}_i)^2 \sigma(a\bar{x}_2 \bar{x}_i)^2 \\ \frac{Z_{\mathrm{UO}}(n;\vec{x})}{Z_{\mathrm{UO}}(n-1;\vec{x}'')} &= b^2 \sigma(a)^2 \sigma(a^2)^2 \sigma(a^2 x_1^2) \sigma(a^2 x_2^2) \\ & \cdot \sigma(cx_1) \sigma(cx_2) \prod_{3 \leq i \leq 2n} \sigma(a\bar{x}_1 \bar{x}_i)^2 \sigma(a\bar{x}_2 \bar{x}_i)^2 \\ & \cdot \prod_{3 \leq i \leq 2n} \sigma(a\bar{x}_1 x_i)^2 \sigma(a\bar{x}_2 x_i)^2. \end{split}$$

Proof. This lemma is clearer in the alternating-sign matrix model than it is in the square ice model. The partition function $Z(n; \vec{x}, \vec{y})$ is a sum over $n \times n$ alternating-sign matrices in which each entry of the matrix has a multiplicative weight. When $y_1 = ax_1$, the weight of a 0 in the southwest corner is 0. Consequently this corner is forced to be 1 and the left column and bottom row are forced to be 0, as in Figure 16. The

$$\begin{pmatrix}
0 & 0 & 0 & + & 0 \\
0 & + & 0 & - & + \\
0 & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & 0 \\
+ & 0 & 0 & 0 & 0
\end{pmatrix}$$

Figure 16: ASM entries forced by $y_1 = ax_1$.

sum reduces to one over $(n-1) \times (n-1)$ ASMs. The only discrepancy between $Z(n; \vec{x}, \vec{y})|_{y_1 = ax_1}$ and $Z(n-1; \vec{x}', \vec{y}')$ is the weights of the forced entries, which the lemma lists as factors.

The argument in the other determinant cases is identical. The argument in the Pfaffian cases is only slightly different: All QTSASMs have zeroes in the corners, and the specialization $x_2 = ax_1$ instead forces a 1 next to each corner and zeroes the first two rows and columns from each edge. Likewise the specialization $x_2 = \bar{a}\bar{x}_1$ forces a 1 next to each corner of an OSASM or an OOSASM and a 1 in the third row entry bottom of a UOSASM, and several rows and columns of zeroes in each of these cases.

Define the *width* of a Laurent polynomial to be the difference in degree between the leading and trailing terms. (For example, $q^3 - q^{-2}$ has width 5.)

Lemma 14. Both sides of each equation of Theorem 10 are Laurent polynomials in each coordinate of \vec{x} (and \vec{y} in the determinant cases) and their widths in x_1 (y_1 in the determinant cases) are as given in Table 1.

To conclude the proof of Theorem 10, we claim that Lemmas 11, 12, and 13 inductively determine both sides by Lagrange interpolation. (To begin the induction each partition function is set to 1 when n = 0.) If a Laurent polynomial of width w has prespecified leading and trailing exponents, it is determined by w+1 of its values. Each of our partition functions is a centered Laurent polynomial in x_1 (in the Pfaffian cases) or y_1 (in the determinant cases). Moreover each is either an even function or an odd function. Thus we only need w+1 specializations, where w is the width in x_1^2 (or y_1^2).

These widths are summarized in Table 1. To compute them, observe that each 0 entry in the bottom row of an ASM contributes 1 to the width. In the UASM and UUASM cases, it is the bottom two rows, and the U-turn itself contributes 1 to the width as well. In the QTSASM case, the corner entries always have weight $\sigma(a)$ and do not contribute to the width. Lemmas 11 and 13 together provide many specializations which are listed in Table 1. Note that Lemma 11 implies that $\sigma(a^2x_1^2)$ divides $Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$ $Z_{\mathrm{UO}}(n;\vec{x},vy)$, which provides an extra specialization in these two cases. In conclusion, it is easy to check that there are enough specializations to match the widths.

Remark. The formulas in Theorem 10 are even more special than Lemmas 11 through 14 suggest. Among the evidence for this, the recurrence relations still hold with only slight modifications if all spectral parameters in the QT, UU, and UO grids

Function	Width	Specializations
$Z(n; \vec{x}, \vec{y})$	n-1	$y_1 = ax_i$
$Z_{ m HT}^{\pm}(n;\vec{x},\vec{y})$	2n - 1	$y_1 = a^{\pm 1} x_i$
$Z_{\mathrm{U}}(n; \vec{x}, \vec{y})$	2n - 1	$y_1 = ax_i^{\pm 1}$
$Z_{\mathrm{UU}}(n;\vec{x},\vec{y})$	4n	$y_1 = a^{\pm 1} x_i^{\pm 1}, a$
$Z_{\mathrm{QT}}(n; \vec{x})$	4n - 3	$x_1 = a^{\pm 1} x_i$
$Z_{\rm O}(n;\vec{x})$	2n-2	$x_1 = a\bar{x}_i$
$Z_{\text{OO}}(n; \vec{x})$	4n - 3	$x_1 = a^{\pm 1}\bar{x}_i$
$Z_{\mathrm{UO}}(n;\vec{x})$	8n - 4	$x_1 = a^{\pm 1} x_i^{\pm 1}, \bar{a}$

Table 1: Widths and specializations of partition functions..

are multiplied by an extra parameter z. Similarly the spectral parameters in the top halves of the HT and OO grids may be multiplied by an arbitrary z instead of by ± 1 . However, we were not able to generalize Theorem 10 to include this parameter.

Lemma 13 reveals another subtlety, namely that

$$\frac{Z_{\rm HT}^+(n; \vec{x}, \vec{y})}{Z_{\rm HT}^+(n-1; \vec{x}', \vec{y}')} = \left(\frac{Z(n; \vec{x}, \vec{y})}{Z(n-1; \vec{x}', \vec{y}')}\right)^2$$

at every specialization $y_i = a^{\pm 1}x_j$. Since this coincidence holds for enough specializations to determine $Z_{\rm HT}^+(n;\vec x,\vec y)$ entirely, one might suppose that

$$Z_{\mathrm{HT}}^{+}(n;\vec{x},\vec{y}) = Z(n;\vec{x},\vec{y})^{2}.$$

But then $Z_{\rm HT}^+(n;\vec{x},\vec{y})$ would be an even function of y_1 , while in reality it is an odd function. The other symmetry classes involving half-turn rotation have similar behavior.

6. FACTOR EXHAUSTION

In this section we derive several round determinants and Pfaffians depending on two and three parameters. We will later identify special cases of the determinants and Pfaffians with those appearing in Theorem 10, and they will specialize further to establish the enumerations in Theorems 2, 3, and 5.

Since the formulas in this section may seem complicated, we recommend verifying that they are round without worrying about their exact form in the first reading. For this purpose we give a more precise definition of roundness that also applies to polynomials. A term R_n in a sequence of rational polynomials depending on one or more variables is *round* if it is a ratio of products of constants, monomials, and differences of two monic terms. All exponents and constant factors should grow polynomially in n or be independent of n. For example, $n!3^n(q+p^n)$ is round and Gaussian binomial coefficients are round. Note that a round polynomial in a single variable must be a product of cyclotomic polynomials, which is part of the motivation for the term "round". Roundness is preserved when a variable is set to 1 or to a product of other variables. In a later reading one can verify the explicit formulas. This is

a tedious but elementary computation, because all round expressions involved have an explicit and regular form. As a warmup the reader can verify that the expressions for A(n) in Theorems 1 and 2 coincide.

We begin with the classic Cauchy double alternant and a Pfaffian generalization found independently by Stembridge and by Laksov, Lascoux, and Thorup [16, 24, 37, 38].

Theorem 15 (Cauchy, S. L. L. T.). Let

$$\begin{split} C_1(\vec{x}, \vec{y})_{i,j} &= \frac{1}{x_i + y_j} \\ C_2(\vec{x}, \vec{y})_{i,j} &= \frac{1}{x_i + y_j} - \frac{1}{1 + x_i y_j} \end{split}$$

For $1 \le i, j \le 2n$, let

$$C_3(\vec{x})_{i,j} = \frac{x_j - x_i}{x_i + x_j}$$

$$C_4(\vec{x})_{i,j} = \frac{x_j - x_i}{1 - x_i x_j}.$$

Then

$$\begin{split} \det C_1 &= \frac{\prod_{i < j} (x_j - x_i) (y_j - y_i)}{\prod_{i,j} (x_i + y_j)} \\ \det C_2 &= \frac{\prod_{i < j} (1 - x_i x_j) (1 - y_i y_j) (x_j - x_i) (y_j - y_i)}{\prod_{i,j} (x_i + y_j) (1 + x_i y_j)} \\ & \cdot \prod_i (1 + x_i) (1 + y_i) \end{split}$$

$$\text{Pf} C_3 &= \prod_{i < j \le 2n} \frac{x_i - x_j}{x_i + x_j} \\ \text{Pf} C_4 &= \prod_{i < i < 2n} \frac{x_i - x_j}{1 - x_i x_i} \end{split}$$

Proof. Our proof is by the factor exhaustion method [20]. The determinant $\det C_1$ is divisible by $x_j - x_i$ because when $x_i = x_j$, two rows of C_1 are proportional. Likewise it is also divisible by $y_i - y_i$. At the same time, the polynomial

$$\prod_{i,j} (x_i + y_j)(\det C_1)$$

has degree $n^2 - n$, so it has no room for other non-constant factors. This determines $\det C_1$ up to a constant, which can be found inductively by setting $x_1 = -y_1$.

The determinant $\det C_2$ is argued the same way. The Pfaffians $\operatorname{Pf} C_3$ and $\operatorname{Pf} C_4$ are also argued the same way; here the constant factor can be found by setting $x_1 = \bar{x}_2$.

Next we evaluate four determinants in the variables p and q. We use two more functions similar to σ and α from Section 5:

$$\gamma(q) = q^{1/2} - q^{-1/2} \qquad \tau(q) = q^{1/2} + q^{-1/2}.$$

Theorem 16. Let

$$\begin{split} T_{1}(p,q)_{i,j} &= \frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} \\ T_{2}(p,q)_{i,j} &= \frac{\tau(q^{j-i})}{\tau(p^{j-i})} \\ T_{3}(p,q)_{i,j} &= \frac{\gamma(q^{n+j+i})}{\gamma(p^{n+j+i})} - \frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} \\ T_{4}(p,q)_{i,j} &= \frac{\tau(q^{j+i})}{\tau(p^{j+i})} - \frac{\tau(q^{j-i})}{\tau(p^{j-i})} \end{split}$$

Then

$$\det T_1 = \frac{\prod_{i \neq j} \gamma(p^{j-i}) \prod_{i,j} \gamma(qp^{j-i})}{\prod_{i,j} \gamma(p^{n+j-i})}$$

$$2^n \prod_{i \neq j} \gamma(p^{j-i})^2 \prod_{\substack{i,j \\ 2 \nmid j-i}} \gamma(qp^{j-i})$$

$$\det T_2 = (-1)^{\binom{n}{2}} \frac{2^{|j-i|}}{\prod_{\substack{i,j \\ 1,j \\ 2 \mid j-i}} \tau(p^{j-i})}$$

$$\det T_3 = \frac{\prod_{\substack{i < j \leq 2n \\ 2 \mid j}} \gamma(p^{j-i}) \prod_{\substack{i,j \leq 2n+1 \\ 2 \mid j}} \gamma(qp^{j-i})}{\prod_{\substack{i,j \\ 2 \mid j}} \gamma(p^{n+j-i}) \gamma(p^{n+j+i})}$$

$$\det T_4 = \frac{2^n \prod_{\substack{i < j \leq n \\ 2 \nmid i, 2 \mid j}} \gamma(p^{2(j-i)})^2 \prod_{\substack{i,j \leq 2n+1 \\ 2 \nmid i, 2 \mid j}} \gamma(qp^{j-i})}{\prod_{\substack{i,j \\ 1,j \\ 2 \mid i, 2 \mid j}} \tau(p^{j-i}) \tau(p^{j+i})}$$

Proof. Factor exhaustion. We first view each determinant as a fractional Laurent polynomial in q. By choosing special values of q, we will find enough factors in each determinant to account for their entire width, thus determining them up to a rational factor R(p). (Each determinant is a centered Laurent polynomial in q with fractional exponents. The notion of width make sense for these.) We will derive this factor by a separate method.

For example, if $0 \le k < n$, then $\det T_1$ is divisible by $\gamma (qp^{-k})^{n-k}$ because

$$T_1(p, p^k)_{i,j} = \sum_{\substack{\frac{1-k}{2} \le \ell \le \frac{k-1}{2}}} p^{\ell(n+j-i)}.$$

Evidently $T_1(p,p^k)$ is a sum of k rank 1 matrices at this specialization, so its determinant has an (n-k)-fold root at $q=p^k$. Likewise $T(p,p^{-k})$ also has rank k and $\gamma(qp^k)^{n-k}$ also divides T_1 . All four of the determinants have this behavior. In each case, the singular values of q can be read from the product formulas for the determinants. The only detail that changes is the form of each rank 1 term, which is summarized in Table 2.

Finally the q-independent factor R(p) can be found by examining the coefficient of the leading power of q, or equivalently, taking the limit $q \to \infty$. For example

$$\frac{T_1(p,q)_{i,j}}{q^{(n+j-i)/2}} \to \frac{1}{\gamma(p^{n+j-i})} = p^{(n+i+j)/2}C(\vec{x}, \vec{y})$$

as $q \rightarrow \infty$ with

$$x_i = p^{-i}$$
 $y_j = p^{n+j}$.

Matrix	Rank 1 terms	Extra q value
T_1	$p^{-\ell i}p^{\ell(n+j)}$	∞
T_2	$p^{-\ell i} p^{\ell j}$	1
T_3	$(p^{-\ell i} - p^{\ell i})(p^{\ell(n+j)} - p^{-\ell(n+j)})$	∞
T_4	$(p^{-\ell i}-p^{\ell i})(p^{\ell j}-p^{-\ell j})$	∞

Table 2: Details of factor exhaustion for Theorem 16.

In this case R(p) is given by $\det C_1$ in Theorem 15. This happens in each case, although for the matrix T_2 it is slightly more convenient to specialize to q=1. The best extra value of q in all four cases is given in Table 2.

Finally we evaluate two three-variable Pfaffians which are like the determinants in Theorem 16.

Theorem 17. For $i, j \leq 2n$, let

$$\begin{split} T_5(p,q,r)_{i,j} &= \frac{\gamma(q^{j-i})\gamma(r^{j-i})}{\gamma(p^{j-i})} \\ T_6(p,q,r)_{i,j} &= \gamma(p^{j+i})\gamma(p^{j-i}) \bigg(\frac{\gamma(q^{j+i})}{\gamma(p^{j+i})} - \frac{\gamma(q^{j-i})}{\gamma(p^{j-i})} \bigg) \\ &\qquad \bigg(\frac{\gamma(r^{j+i})}{\gamma(p^{j+i})} - \frac{\gamma(r^{j-i})}{\gamma(p^{j-i})} \bigg). \end{split}$$

when $i \neq j$ and

$$T_5(p,q,r)_{i,i} = 0$$

 $T_6(p,q,r)_{i,i} = 0$

Then

$$\begin{split} \operatorname{Pf} T_5 &= \frac{\prod_{i < j} \gamma(p^{j-i})^4 \prod_{i,j} \gamma(qp^{j-i}) \gamma(rp^{j-i})}{\prod_{i < j \leq 2n} \gamma(p^{j-i})} \\ \operatorname{Pf} T_6 &= \frac{\prod_{i < j \leq 2n} \gamma(p^{j-i}) \prod_{i,j \leq 2n+1} \gamma(qp^{j-i}) \gamma(rp^{j-i})}{\prod_{i < j < 2n} \gamma(p^{j+i})}. \end{split}$$

Proof. Factor exhaustion in both q and r. If $0 \le k < n$, then

$$T_5(p,p^k,r)_{i,j} = \sum_{\frac{1-k}{2} \leq \ell \leq \frac{k-1}{2}} r^{1/2} p^{\ell(j-i)} - \sum_{\frac{1-k}{2} \leq \ell \leq \frac{k-1}{2}} r^{-1/2} p^{\ell(j-i)}$$

is, as written, a sum of 2k rank 1 matrices. Therefore the Pfaffian, whose square is the determinant, it is divisible by $\gamma(qp^{-k})^{n-k}$. The same argument applies to $T_5(p,p^{-k},r)$. It also applies to $T_5(p,q,p^{\pm k})$ since T_5 is symmetric in q and r.

This determines Pf T_5 up to a factor R(p) depending only on p. This factor can be determined by taking the limit $r \to \infty$:

$$\lim_{r \to \infty} \frac{T_5(p,q,r)_{i,j}}{r^{(|i-n-\frac{1}{2}|+|j-n-\frac{1}{2}|)/2}} = \begin{cases} T_1(p,q)_{i,j-n} & i \le n < j \\ -T_1(p,q)_{j,i-n} & j \le n < i \end{cases}.$$

In other words, after rescaling rows and columns, $T_5(p,q,r)$ has a block matrix limit:

$$\lim_{r \to \infty} r^{\bullet} T_5(p,q,r) = \left(\begin{array}{c|c} 0 & T_1(p,q) \\ \hline -T_1(p,q)^T & 0 \end{array} \right).$$

(The bullet \bullet is the exponent above that is different for different rows and columns.) This establishes that the leading coefficient of Pf $T_5(p,q,r)$ as a polynomial in r is

$$(-1)^{\binom{n}{2}} \det T_1(p,q),$$

which in turn determines R(p).

The Pfaffian Pf T_6 is argued the same way. To find the factor R(p) which is independent of q and r, we take the limit $r, q \to \infty$. In this limit Pf T_6 reduces to a special case of Pf C_4 in Theorem 15.

Remark. Several other specializations of the determinants in Theorem 16 and the Pfaffian in Theorem 17 are special cases of Theorem 15 and other determinants and Pfaffians such as these [20]:

$$\det\left\{x_i^{j-1}\right\} \qquad \det\left\{\gamma(x_i^{2j-1})\right\}$$

For example, $T_1(q^2,q)$ is also a Cauchy double alternant, while $T_1(p,p^n)$ is the product of two (rescaled) Vandermonde matrices. Any of these intersections may be used to determine the q-independent factor in the factor exhaustion method. There are also other round determinants like the ones in Theorem 16 which we do not need, for example

$$\det\left\{\frac{\gamma(q^{n+j-i})}{\gamma(p^{n+j-i})} - \frac{\gamma(q^{n+j+i-1})}{\gamma(p^{n+j+i-1})}\right\}.$$

These examples suggest the following more general problem: Let M be an $n \times n$ matrix such that $M_{i,j}$ is a rational polynomial in a fixed number of variables, such as p, q, and r, and in exponentials of them such as p^i , q^j , and r^n . When is det M round? What if M_{ij} is a rational polynomial in variables such as x_i and y_i ?

7. ENUMERATIONS AND DIVISIBILITIES

In this section we relate the quantities appearing in the other sections to prove the results in Section 1.

Let

$$\vec{1} = (1, 1, 1, 1, \dots, 1)$$

$$x = a^2 + 2 + \bar{a}^2$$

$$y = \sigma(ba) / \sigma(b\bar{a})$$

$$z = \sigma(ca) / \sigma(c\bar{a}).$$
(1)

Then most of the generating functions in Section 1 can be

expressed in terms of the partition functions in Section 5,

$$A(n;x) = \frac{Z(n;\vec{1},\vec{1})}{\sigma(a)^{n^{2}-n}\sigma(a^{2})^{n}}$$
(2)
$$A_{HT}(2n;x,\pm 1) = \frac{Z_{HT}^{\pm}(n;\vec{1},\vec{1})}{\sigma(a)^{2n^{2}-n}\sigma(a^{2})^{n}}$$

$$A_{U}(2n;x,y) = \frac{Z_{U}(n;\vec{1},\vec{1})}{\sigma(a)^{2n^{2}-n}\sigma(a^{2})^{n}\sigma(b\bar{a})^{n}}$$

$$A_{UU}(4n;x,y,z) = \frac{Z_{UU}(n;\vec{1},\vec{1})}{\sigma(a)^{4n^{2}-n}\sigma(a^{2})^{n}\sigma(b\bar{a})^{n}\sigma(c\bar{a})^{n}}$$

$$A_{QT}(4n;x) = \frac{Z_{QT}(n;\vec{1})}{\sigma(a)^{4n^{2}-n}\sigma(a^{2})^{n}}$$

$$A_{O}(2n;x) = \frac{Z_{O}(n;\vec{1})}{\sigma(a)^{2n^{2}-2n}\sigma(a^{2})^{n}}$$

$$A_{UO}(8n;x,z) = \frac{Z_{UO}(n;\vec{1},\vec{1})}{\sigma(a)^{8n^{2}-3n}\sigma(a^{2})^{n}\sigma(c\bar{a})^{n}b^{2n}},$$

by the definition of the partition functions and the correspondence between square ice and alternating-sign matrices. The generating function $A_{\rm OO}(4n;x,y)$ requires a slightly different change of parameters: if $y=b^2/c^2$, then

$$A_{OO}(4n; x, y) = \frac{Z_{OO}(n; \vec{1})}{\sigma(a)^{4n^2 - 3n} \sigma(a^2)^n c^{2n}}.$$

The generating function $A_{\rm U}(n;x,y)$ is a polynomial of degree n in y and it is easy to show that the leading and trailing coefficients count VSASMs, so we can say that

$$A_{\rm V}(2n+1;x) = A_{\rm II}(n;x,0) = A_{\rm II}(n;x,\infty),$$

where by abuse of notation, if P(x) is a polynomial (or a rational function), $P(\infty)$ denotes the top-degree coefficient. Likewise $A_{\mathrm{UU}}(n;x,y,z)$ has bidegree (n,n) in y and z and the corner coefficients count VHSASMs and VHPASMs:

$$\begin{split} A_{\text{VHP}}(4n+2;x) &= A_{\text{UU}}(n;x,0,\infty) = A_{\text{UU}}(n;x,\infty,0) \\ A_{\text{VH}}(4n+1;x) &= A_{\text{UU}}(n;x,\infty,\infty) \\ A_{\text{VH}}(4n+3;x) &= A_{\text{UU}}(n;x,0,0). \end{split}$$

We can reverse these relations by defining

$$\begin{split} Z_{\text{HT}}^{\pm,(2)}(n;\vec{x},\vec{y}) &= \frac{\prod_{i,j} \alpha(x_i \bar{y}_j) (\det M_{\text{HT}}^{\pm})}{\prod_{i < j} \sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j)} \\ Z_{\text{UU}}^{(2)}(n;\vec{x},\vec{y}) &= \frac{\prod_{i,j} \alpha(x_i \bar{y}_j) \alpha(x_i y_j) (\det M_{\text{UU}})}{\prod_{i < j} \sigma(\bar{x}_i x_j) \sigma(y_i \bar{y}_j) \prod_{i \leq j} \sigma(\bar{x}_i \bar{x}_j) \sigma(y_i y_j)} \\ Z_{\text{QT}}^{(k)}(n;\vec{x}) &= \frac{\prod_{i < j \leq 2n} \alpha(\bar{x}_i x_j) (\text{Pf} M_{\text{QT}}^{(k)})}{\prod_{i < j \leq 2n} \sigma(\bar{x}_i x_j)} \\ Z_{\text{OO}}^{(2)}(n;\vec{x}) &= \frac{\prod_{i < j \leq 2n} \alpha(x_i x_j) (\text{Pf} M_{\text{OO}})}{\prod_{i < j \leq 2n} \sigma(\bar{x}_i x_j)} \\ Z_{\text{UO}}^{(k)}(n;\vec{x}) &= \frac{\prod_{i < j \leq 2n} \alpha(\bar{x}_i x_j) \alpha(x_i x_j) (\text{Pf} M_{\text{UO}}^{(k)})}{\prod_{i < j \leq 2n} \sigma(\bar{x}_i x_j) \prod_{i < j \leq 2n} \sigma(x_i x_j)} \end{split}$$

and

$$\begin{split} A_{\mathrm{HT}}^{(2)}(2n;x,\pm 1) &= \sigma(a)^{n-n^2} Z_{\mathrm{HT}}^{\pm,(2)}(2n;\vec{1},\vec{1}) \\ A_{\mathrm{UU}}^{(2)}(4n;x,y,z) &= \sigma(a)^{n-2n^2} \sigma(b\bar{a})^{-n} \sigma(c\bar{a})^{-n} Z_{\mathrm{UU}}^{(2)}(4n;\vec{1},\vec{1}) \\ A_{\mathrm{QT}}^{(k)}(4n;x) &= \sigma(a)^{n-2n^2} Z_{\mathrm{QT}}^{(k)}(4n;\vec{1}) \\ A_{\mathrm{OO}}^{(2)}(4n;x,y) &= c^{-2n} \sigma(a)^{2n-2n^2} Z_{\mathrm{OO}}^{(2)}(4n;\vec{1}) \\ A_{\mathrm{UO}}^{(1)}(8n;x,z) &= \sigma(a)^{n-4n^2} Z_{\mathrm{UO}}^{(1)}(8n;\vec{1},\vec{1}) \\ A_{\mathrm{UO}}^{(2)}(8n;x,z) &= \sigma(a)^{n-4n^2} \sigma(c\bar{a})^{-n} Z_{\mathrm{UO}}^{(2)}(8n;\vec{1},\vec{1}). \end{split}$$

using the same correspondence between a,b, and c with x,y, and z (which slightly different in the case of OOSASMs). The observation that all of these quantities must be polynomials establishes the factorizations of $A_{\rm HT},A_{\rm OO},A_{\rm QT},A_{\rm UU}$, and $A_{\rm UO}$ in Theorem 4.

For vectors \vec{x} and \vec{y} , let

$$(\vec{x},\vec{y})=(x_1,\ldots,x_n,y_1,\ldots,y_n)$$

denote their concatenation, and let exponentiation of vectors denote coordinate-wise exponentiation:

$$\vec{x}^k = (x_1^k, x_2^k, \dots, x_n^k).$$

Then the matrices

$$\begin{split} &M(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1}))\\ &M_{\mathrm{HT}}(2n;(\vec{x},\vec{x}^{-1}),(\vec{y},\vec{y}^{-1})) \end{split}$$

commutes with the permutation matrix

$$\left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right)$$
,

where I_n is the $n \times n$ identity matrix. Similarly the matrices

$$M(2n+1;(\vec{x},1,\vec{x}^{-1}),(\vec{y},1,\vec{y}^{-1}))$$

 $M_{\rm HT}(2n+1;(\vec{x},1,\vec{x}^{-1}),(\vec{y},1,\vec{y}^{-1}))$

commute with

$$P = \begin{pmatrix} 0 & 0 & I_n \\ \hline 0 & 1 & 0 \\ \hline I_n & 0 & 0 \end{pmatrix}.$$

If \vec{x} has length 2n and $b^2 = -c^2$, then

$$M_{\text{OO}}(4n;(\vec{x},\vec{x}^{-1}))$$

commutes with

$$P = \left(\begin{array}{c|c} 0 & I_{2n} \\ \hline I_{2n} & 0 \end{array}\right).$$

In each case we can restrict decompose M, $M_{\rm HT}$, and $M_{\rm OO}$ into blocks corresponding to the eigenspaces of P. The block

with eigenvalue -1 is in the three cases equal to $M_{\rm U}$ and proportional to $M_{\rm UU}$ (with $b=c={\bf i}$) and $M_{\rm UO}^{(1)}$. This results in the factorizations of A(n;x), $A_{\rm HT}^{(2)}(n;x,1)$, and $A_{\rm OO}^{(2)}(4n;x,-1)$ in Theorem 4. Another factorization in Theorem 4 is that of $A_{\rm HT}^{(2)}(2n;x,-1)$. To establish this, observe that the matrix $M_{\rm HT}^{-}(2n;\vec{x},\vec{x})$ is antisymmetric, and that it is proportional to $M_{\rm QT}^{(1)}(2n;\vec{x})$. The latter matrix is employed for its Pfaffian, while the former for its determinant, which is the square of the Pfaffian.

The final case of Theorem 4 is the relation between $A_{\rm V}(2n+1;x)$ and $A_{\rm U}(2n;x)$. This relation is established by observing that b appears only in the normalization factor for $Z_{\rm U}(n;x)$ and not in the matrix $M_{\rm U}(n;x)$; the only step is to change variables from b to y.

Enumeration	Parameters	Section 5	Section 6
A(n;1)	$a = \omega_3$	$M(\vec{q},\vec{q}(n))$	$T_1(q^3,q)$
A(n;2)	$a = \omega_4$	$M(\vec{x}, \vec{y})$	$C_1(\vec{x}^2, \vec{y}^2)$
A(n;3)	$a = \omega_6$	$M(\vec{q},\vec{q})$	$T_2(q^3,q)$
$A_{\rm HT}(2n;1,1)$	$a = \omega_3$	$M^+_{ m HT}(\vec{q},\vec{q}(n))$	$T_1(q^3, q^2)$
$A_{\rm HT}(2n;2,1)$	$a = \omega_4$	$M^+_{ m HT}(\vec{q},\vec{q})$	$T_2(q^2,q)$
$A_{\mathbf{V}}(2n+1;1)$	$a = \omega_3$	$M_{\mathrm{U}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q)$
$A_{\mathbf{V}}(2n+1;2)$	$a = \omega_4$	$M_{\mathrm{U}}(\vec{x}, \vec{y})$	$C_2(\vec{x}^2, \vec{y}^2)$
$A_{\mathbf{V}}(2n+1;3)$	$a = \omega_6$	$M_{\mathrm{U}}(\vec{q},\vec{q})$	$T_4(q^3,q)$
$A_{\mathrm{UU}}^{(2)}(4n;1,1,1)$	$a = \omega_3, b = c = \omega_4$	$M_{\mathrm{UU}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q^2)$
$A_{\text{UU}}^{(2)}(4n;2,1,1)$	$a = b = c = \omega_4$	$M_{\mathrm{UU}}(\vec{q},\vec{q})$	$T_4(q^2,q)$
$A_{\text{VHP}}^{(2)}(4n;1)$	$a = b = \omega_3, c = \bar{a}$	$M_{\mathrm{UU}}(\vec{q},\vec{q}(n))$	$T_3(q^3,q)$
$A_{\mathrm{QT}}^{(1)}(4n;1)$	$a = \omega_3$	$M_{\mathrm{QT}}^{(1)}(\vec{q})$	$T_5(q^3,q,q)$
$A_{ m OT}^{(1)}(4n;2)$	$a = \omega_4$	$M_{ m OT}^{(1)}(ec{q})$	$T_5(q^4, q^2, q)$
$A_{\text{OT}}^{(1)}(4n;3)$	$a = \omega_6$	$M_{ m OT}^{(1)}(ec{q})$	$T_5(q^6,q^3,q^2)$
$A_{\mathrm{OT}}^{(2)}(4n;1)$	$a = \omega_3$	$M_{ m OT}^{(2)}(\vec{q})$	$T_5(q^3,q^2,q)$
$A_{ m OT}^{(2)}(4n;2)$	$a = \omega_4$	$M_{\mathrm{OT}}^{(2)}(\vec{x})$	$C_3(\vec{x}^2)$
$A_{O}(2n;1)$	$a = \omega_3$	$M_{\Omega}(\vec{q})$	$T_6(q^3, q, \infty)$
$A_{\text{LIO}}^{(1)}(8n;1)$	$a = \omega_3, c = \omega_4$	$M_{\mathrm{UO}}^{(1)}(\vec{q})$	$T_6(q^3,q,q)$
$A_{\rm UO}^{(2)}(8n;1,1)$	$a = \omega_3, c = \omega_4$	$M_{\mathrm{UO}}^{(2)}(\vec{q})$	$T_6(q^3, q^2, q)$

Table 3: Specializations of partition function determinants and Pfaffians.

Finally we establish the round enumerations in Theorems 2, 3, and 5. We review the argument from Reference 23 for A(n), which is an illustrative case. Let $\omega_n = \exp(\pi \mathbf{i}/n)$, where $\mathbf{i}^2 = -1$. Equation 1 implies the following correspondence between x and a:

$$a = \omega_3 \implies x = 1$$

 $a = \omega_4 \implies x = 2$
 $a = \omega_6 \implies x = 3$.

For any of these values of x or a, we would like to evaluate the partition function $Z(n; \vec{1}, \vec{1})$ to find A(n; x) by equation 2. Unfortunately the matrix $M(n; \vec{1}, \vec{1})$ is singular. So instead we

will find its determinant along a curve of parameters that includes $(\vec{1}, \vec{1})$. More precisely, let

$$\vec{q}(k) = (q^{(k+1)/2}, q^{(k+2)/2}, \dots, q^{(k+n)/2})$$

and

$$\vec{q} = \vec{q}(0).$$

Then

$$\lim_{q\to 1} \vec{q}(k) = \vec{1},$$

and if we assume equation 1,

$$M(n;\vec{q}(0),\vec{q}(k)) = \frac{1}{-q^{k+j-i} + x - 2 - q^{i-j-k}}.$$

If we further set x = 1 and k = n, then

$$M(n; \vec{q}(0), \vec{q}(n)) = -T_1(q^3, q)$$

has a round determinant by Theorem 16. Computing A(n;x) is then a routine but tedious simplification of round products. The argument for most of the other enumerations is the same, except that the curve of parameters is $(\vec{q}, \vec{q}(n))$ for 1-enumeration in the determinant cases, (\vec{q}, \vec{q}) for 2- and 3-enumeration in the determinant cases, and \vec{q} in the Pfaffian cases. Each of the matrices is then proportional to some matrix T_i from Theorem 16 or 17. The determinants and Pfaffian for three of the 2-enumerations are round without specializing the parameters and instead reduce to a matrix C_i from Theorem 15. These variations of the argument are summarized in Table 3.

8. DISCUSSION

Even though each section of this article considers many types of alternating-sign matrices or determinants in parallel, the work of enumerating symmetry classes of ASMs is far from finished. Robbins [31] conjectures formulas for the number of VHSASMs and for the number of odd-order HTSASMs, QTSASMs, and DASASMs in addition to the enumerations that we have proven. Theorem 10 yields a determinant formula for the number of VHSASMs, obtained from the more general partition function $Z_{\text{UU}}(n;\vec{x},\vec{y})$ by setting $a=\omega_3$ and $b=c=\omega_3^{\pm 1}$, and all other parameters to 1). In the enumeration of $4n+1\times 4n+1$ VHSASMs, where $b=c=\omega_3$, experiments indicate that

$$\det M_{\mathrm{UU}}(\vec{q}(-\frac{1}{2}), \vec{q}(n-\frac{1}{2}))$$

is round, but we cannot prove this. In the other case, $4n+3 \times 4n+3$ VHSASMs, where $b=c=\omega^{-1}$, we could not even find a curve for which the determinant is round. This strange behavior of VHSASMs is one illustration that although we have put many classes of ASMs under one roof, the house is not completely in order.

For the other three classes conjecturally enumerated by Robbins, we could not even find a determinant formula. Nonetheless we conjecture:

Factor	n = 1	n = 2	n = 3
$A_{\mathbf{V}}(2n+1;x)$	1	x+2	$x^3 + 6x^2 + 13x + 6$
$\widetilde{A}_{V}(2n;x)$	1	x+6	$x^3 + 12x^2 + 70x + 60$
$A_{\text{OT}}^{(1)}(4n;x)$	1	x+3	$x^3 + 8x^2 + 25x + 15$
$A_{\text{OT}}^{(2)}(4n;x)$	2	6x + 4	$20x^3 + 60x^2 + 52x + 8$
	x+4	$x^3 + 9x^2 + 40x + 16$	$x^6 + 16x^5 + 125x^4 + 629x^3 + 1036x^2 + 560x + 64$
$\widetilde{A}_{\mathrm{UU}}^{(2)}(4n;x)$	1	x+1	$x^3 + 4x^2 + 5x + 1$
$A_{\text{VHP}}^{(2)}(4n+2;x)$	1	2x + 1	$5x^3 + 12x^2 + 8x + 1$
	1	3	$x^2 + 10x + 15$
$A_{\text{OO}}^{(2)}(4n;x,1)$	2	2x + 4	$2x^3 + 20x^2 + 28x + 8$
	1	$x^2 + 5x + 3$	$x^6 + 14x^5 + 82x^4 + 210x^3 + 239x^2 + 115x + 15$
	2	$2x^2 + 20x + 20$	$2x^6 + 42x^5 + 420x^4 + 1680x^3 + 2892x^2 + 2040x + 360$
$A_{\text{UO}}^{(2)}(8n;x,1)$	x+4	$x^4 + 12x^3 + 65x^2 + 104x + 16$	$x^9 + 24x^8 + 275x^7 + 1966x^6 + 8215x^5 + 19144x^4$
			$+21777x^3 + 10028x^2 + 1712x + 64$

Table 4: Irreducible *x*-enumerations

Question 18. Can DSASMs, DASASMs, TSASMs, and oddorder HTSASMs and QTSASMs can be x-enumerated in polynomial time?

The polynomials listed in Table 4 appear to be generating functions of some type, but in most cases there is not even a proof that their coefficients are non-negative. (We have more data than is shown in the table; the multivariate polynomials $A_{\rm UU}^{(2)}(4n;x,y,z), A_{\rm OO}^{(2)}(4n;x,y)$, and $A_{\rm UO}^{(2)}(4n;x,y)$ also appear to be non-negative.) Some of them are conjecturally related to cyclically symmetric plane partitions [31]. Indeed they are related to each other in strange ways. For example Theorem 5 establishes that if we take x=1, three of the polynomial series $(A_{\rm V},A_{\rm O},$ and $A_{\rm VHP}^{(2)})$ become equal, as if to suggest that VSASMs can be x-enumerated in three different ways!

Question 19. Do the polynomials in Table 4 x-enumerate classes of alternating-sign matrices?

OSASMs include the set of off-diagonal permutation matrices, which can be interpreted as the index set for the usual combinatorial formula for the Pfaffian. Like ASMs, their number is round. These observations, together with the known formulas due to Mills, Robbins, and Rumsey [27] motivate the following question:

Question 20. Are there formulas for the Pfaffian of a matrix involving OSASMs that generalize the determinant formulas involving ASMs?

Neither any of the enumerations that we establish, nor the various equinumerations that they imply, have known bijec-

tive proofs. Nor is it even known that two equinumerous types of ASMs index bases of the same vector space. For example, can one find an explicit isomorphism between the vector space of formal linear combinations of $2n \times 2n$ OSASMs and the vector space of formal linear combinations of $2n + 1 \times 2n + 1$ VSASMs?

Sogo found that $Z(n; \vec{1}, \vec{1})$ satisfies the Toda chain (or Toda molecule) differential hierarchy [17, 36].

Question 21. If \vec{x} and \vec{y} are set to $\vec{1}$, do the partition functions in Theorem 10 and Table 4 satisfy natural differential hierarchies?

Many other solutions to the Yang-Baxter equation are known [8]. The six-vertex solution corresponds to the Lie algebra $\mathfrak{sl}(2)$ together with its 2-dimensional representation; there are solutions for other simple Lie algebras and their representations.

Question 22. Do square ice and Izergin-Korepin-type determinants generalize to other solutions of the Yang-Baxter equation?

Although our simultaneous treatment of several classes of ASMs is not especially short, the argument for any one alone is relatively simple. We speculate that the methods of Lagrange interpolation (used in Section 5) and factor exhaustion (the topic of Section 6) simplify many proofs of product formulas. I. J. Good's short proof of Dyson's conjecture [11] also uses Lagrange interpolation.

David M. Bressoud, *Proofs and confirmations*, Mathematical Association of America, Washington, DC, 1999.

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